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A Variation-Diminishing Generalized Spline Approximation Method[1]

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SECTION 1. INTRODUCTION

In this paper we will define a generalized-spline approximation method which is variation-diminishing and preserves functions which are linear in a generalized sense. (Variation-diminishing transformations are defined in Definition 1.3.) In Section 3 we will define an approximating spline with an infinite number of knots. The modifications needed to prove similar results for splines defined on a finite interval with a finite number of knots are indicated in Section 5. Our results generalize the work of Schoenberg reported in [*6*], where he stated without proofs corresponding results for the special case of polynomial splines.

Our generalized splines are piecewise solutions of $\mathscr{M}_n u = 0$, where $\mathscr{M}_n$ is a differential expression of the form

$$(\mathscr{M}_n u)(x) = \left(\frac{1}{w_{n+1}(x)}\frac{d}{dx}\frac{1}{w_n(x)}\frac{d}{dx}\frac{1}{w_{n-1}(x)}\cdots\frac{d}{dx}\frac{1}{w_1(x)}\right)u(x) \tag{1.1}$$

with $w_j(x) > 0$ and $w_j(x)$ of continuity class C^n. It will be useful to define

$$(L_0 u)(x) = u(x),$$

$$(L_j u)(x) = \left(\frac{d}{dx}\frac{1}{w_j(x)}\frac{d}{dx}\frac{1}{w_{j-1}(x)}\cdots\frac{d}{dx}\frac{1}{w_1(x)}\right)u(x), \qquad i = 1, 2, \ldots, n. \tag{1.2}$$

A basic set of solutions for $\mathscr{M}_n u = 0$ is

$$\phi_1(x) = w_1(x),$$

$$\phi_2(x) = w_1(x)\int_\alpha^x w_2(t_2)\,dt_2,$$

$$\phi_j(x) = w_1(x)\int_\alpha^x w_2(t_2)\,dt_2\int_\alpha^{t_2} w_3(t_3)\,dt_3\ldots\int_\alpha^{t_{j-1}} w_j(t_j)\,dt_j, \qquad j = 3, 4, \ldots, n \tag{1.3}$$

where α is a fixed point. Actually, by suitable transformations of the dependent and independent variables, we could assume that $\phi_1(x) \equiv 1$ and $\phi_2(x) = x - \alpha$. Note that $(L_{i-1}\phi_j)(\alpha) = \delta_{ij} w_j(\alpha)$, $i, j = 1, 2, \ldots, n$.

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Definition 1.1. $S(x)$ is a *generalized spline* on (a,b) associated with $\mathscr{M}_n$, with simple knots $\{x_j\}$, if $(\mathscr{M}_n S)(x) = 0$ for $x \in (a,b)$, $x \neq x_j$, and $S(x) \in C^{n-2}(a,b)$.

The following notation is useful.

Definition 1.2. Let $f(x)$ be defined on a subset X of the real line. $S^-(f, X)$ is the number of sign changes of $f(x)$ as x traverses X, where zeros of $f(x)$ are not counted as changes in sign (see [3], page 20).

Definition 1.3. A transformation T which maps a family of functions $\mathscr{F}$ defined on X into functions defined on X_1, is called *variation diminishing* if

$$S^-(Tf; X_1) \leqslant S^-(f; X)$$

for all functions f in $\mathscr{F}$.

Variation-diminishing transformations are investigated extensively in [3].

Using these definitions, we can state our basic results.

Theorem 3.3. *Let $\{x_j\}_{j=-\infty}^{\infty}$ satisfy: $x_j < x_{j+1}$ for all j; $\lim_{j\to-\infty} x_j = a$; $\lim_{j\to\infty} x_j = b$ (we allow $a = -\infty$, $b = \infty$). Let f be defined on (a,b) and continuous. We can find splines $N_j(x)$, $-\infty < j < \infty$, associated with $\mathscr{M}_n$, with simple knots $\{x_j\}_{j=-\infty}^{\infty}$, and points z_j, $-\infty < j < \infty$, $a < z_j < z_{j+1} < b$, such that*

$$\phi_i(x) = \sum_{j=-\infty}^{\infty} \phi_i(z_j)\, N_j(x), \qquad i = 1,2; a < x < b, \tag{1.4}$$

$$S^-\left(\sum_{j=-\infty}^{\infty} f(z_j)\, N_j(x); (a,b)\right) \leqslant S^-(f; (a,b)). \tag{1.5}$$

The $N_j(x)$ and z_j are independent of $f(x)$. (The convergence of $\sum_{j=-\infty}^{\infty} f(z_j)\, N_j(x)$ will hold, since, for each x, only a finite number of terms of the sum are distinct from zero.)

Theorem 5.4. *Let $-\infty < a < x_1 < x_2 < \ldots < x_m < b < \infty$. Let f be a continuous function defined in $[a,b]$. We can find splines $N_j(x)$, $j = 1, 2, \ldots, m+n$, associated with $\mathscr{M}_n$ with simple $\{x_j\}_{j=1}^{m}$ and knots of multiplicity n at $x = a$ (see Definition 5.1), and points z_j, $a = z_1 < z_2 < \ldots < z_{m+n} = b$, such that*

$$\phi_i(x) = \sum_{j=1}^{m+n} \phi_i(z_j)\, N_j(x), \qquad i = 1,2; a \leqslant x \leqslant b,$$

$$S^-\left(\sum_{j=1}^{m+n} f(z_j)\, N_j(x); [a,b]\right) \leqslant S^-(f; [a,b]).$$

The $N_j(x)$ and z_j are independent of $f(x)$.

The fact that the spline approximations are variation-diminishing and preserve generalized linear functions implies that they preserve generalized convexity properties, in the following sense: with $S(x;f) = \sum f(z_j) N_j(x)$,

$$S^-(S(x;f) - a_1 \phi_1(x) - a_2 \phi_2(x); (a,b)) = S^-(S(x; f - a_1 \phi_1 - a_2 \phi_2); (a,b)) \leqslant S^-(f - a_1 \phi_1 - a_2 \phi_2; (a,b)).$$

Thus, if f is a generalized convex or concave function on (a,b), so is the approximating spline $S(x;f)$. (Generalized convexity is discussed in [3], Chapter 6.)

Schoenberg announced the analogous results for polynomial splines in [6], i.e., all $w_j(x)$ are constant, so $\mathscr{M}_n = d^n/dx^n$ and $\phi_j(x) = (x-\alpha)^j$. He was able to evaluate the nodes z_j and splines $N_j(x)$ in some special cases and obtain convergence estimates.

Section 2. Background for Splines with Simple Knots

Generalized splines on (a,b) associated with the differential expression $\mathscr{M}_n$ are defined in Definition 1.1. Our results are based on a representation formula for such splines as linear combinations of certain generalized basic spline functions which were introduced by Karlin in [3] in the study of self-adjoint differential expressions of the form (1.1). By modifying that definition, we can consider non-self-adjoint differential expressions.

Let $\hat{\mathscr{M}}_n$ be the formal differential operator adjoint to $\mathscr{M}_n$:

$$\hat{\mathscr{M}}_n u(x) = (-1)^n \left(\frac{1}{w_1(x)} \frac{d}{dx} \frac{1}{w_2(x)} \frac{d}{dx} \frac{1}{w_3(x)} \cdots \frac{d}{dx} \frac{1}{w_{n+1}(x)} \right) u(x). \tag{2.1}$$

Analogously to (1.2) and (1.3), we define

$$(\hat{L}_0 u)(x) = u(x),$$

$$(\hat{L}_j u)(x) = \left(\frac{d}{dx} \frac{1}{w_{n+2-j}(x)} \frac{d}{dx} \frac{1}{w_{n+3-j}(x)} \cdots \frac{d}{dx} \frac{1}{w_{n+1}(x)} \right) u(x), \qquad i = 1,2,\ldots,n \tag{2.2}$$

and

$$\hat{\phi}_1(x) = w_{n+1}(x),$$

$$\hat{\phi}_2(x) = w_{n+1}(x) \int_\alpha^x w_n(t_n)\, dt_n,$$

$$\hat{\phi}_j(x) = w_{n+1}(x) \int_\alpha^x w_n(t_n)\, dt_n \int_\alpha^{t_n} w_{n-1}(t_{n-1})\, dt_{n-1} \cdots \int_\alpha^{t_{n+3-j}} w_{n+2-j}(t_{n+2-j})\, dt_{n+2-j}, \qquad j = 3,4,\ldots,n, \tag{2.3}$$

where α is a fixed point.

$\{\hat{\phi}_j(x)\}_{j=1}^n$ is a basic set of solutions for $\hat{\mathscr{M}}_n u = 0$. The *fundamental solution* associated with $\hat{\mathscr{M}}_n$ is

$$\hat{\phi}_n(x;s) = \begin{cases} 0 & x < s \\ w_1(s)\, w_{n+1}(x) \int_s^x w_n(t_n)\, dt_n \int_s^{t_n} w_{n-1}(t_{n-1})\, dt_{n-1} \cdots \int_s^{t_3} w_2(t_2)\, dt_2, & s < x. \end{cases} \tag{2.4}$$

For fixed x, it is a generalized spline associated with $\hat{\mathscr{M}}_n$, with a simple knot at $s = x$. It is useful to define

$$\begin{aligned} \hat{\phi}_{j,n}(x;s) &= \hat{L}_{j-1}^{(x)} \phi_n(x;s) \\ &= w_1(s)\, w_{n+2-j}(x) \int_s^x w_{n+1-j}(t_{n+1-j})\, dt_{n+1-j} \cdots \int_s^{t_3} w_2(t_2)\, dt_2, \\ &\qquad s < x, \quad j = 2, 3, \ldots, n, \end{aligned} \tag{2.5}$$

$$\hat{\phi}_{j,n}(x) = \hat{\phi}_{j,n}(x;\alpha)/w_1(\alpha) = \hat{L}_{j-1}\, \hat{\phi}_n(x), \qquad j = 2, 3, \ldots, n; \tag{2.6}$$

the differentiations in (2.5) are to be made with respect to x.

We will assume that we deal with splines with simple knots $\{x_j\}_{j=-\infty}^{\infty}$, where $a = \lim_{j \to -\infty} x_j$, $b = \lim_{j \to \infty} x_j$, and $x_j < x_{j+1}$.

DEFINITION 2.1. The basic spline functions (B-splines) for the differential expression $\mathscr{M}_n$ and knots $\{x_j\}_{j=-\infty}^{\infty}$, are

$$M_k(x) = \frac{\begin{vmatrix} \hat{\phi}_1(x_k) & \cdots & \hat{\phi}_n(x_k) & \hat{\phi}_n(x_k;x) \\ \vdots & & \vdots & \vdots \\ \hat{\phi}_1(x_{k+n}) & \cdots & \hat{\phi}_n(x_{k+n}) & \hat{\phi}_n(x_{k+n};x) \end{vmatrix}}{\det \|\hat{\phi}_j(x_i)\|_{i=k,\ldots,k+n;\, j=1,\ldots,n+1}}, \qquad k = \ldots, -1, 0, 1, \ldots. \tag{2.7}$$

In defining $\hat{\phi}_{n+1}(x)$, we can choose $w_{n+2}(x) \equiv 1$.

Theorem 1.1 in Chapter 6 of [*3*] shows that the denominator of $M_k(x)$ is strictly positive. It is easy to show that the definition of $M_k(x)$ is independent of the choice of α; see [*3*], Chapter 10, Section 4. Therefore, $M_k(x)$ is well-defined.

$M_k(x)$ is a generalized nth divided difference of $\hat{\phi}_n(x;s)$; in fact, in the polynomial spline case, it is a constant times the nth divided difference of $\phi_n(x;s) = (x-s)_+^{n-1}$. Note that $M_k(x)$ is a spline associated with $\mathscr{M}_n$, with simple knots at $x = x_k, x_{k+1}, \ldots, x_{k+n}$.

THEOREM 2.1. *Let $S(x)$ be a spline on (a,b) associated with $\mathcal{M}_n$, with simple knots $\{x_j\}_{j=-\infty}^{\infty}$. Then $S(x)$ can be represented uniquely in the form*

$$S(x) = \sum_{j=-\infty}^{\infty} c_j M_j(x), \tag{2.8}$$

where the c_j are constants.

Proof. See [3], Chapter 10, Section 4.

The sum in (2.8) converges since, for any x, only a finite number of $M_j(x)$ are nonzero. Indeed, Lemma 4.1 in Chapter 10 of [3] shows that $M_j(x) \geqslant 0$ for all x, and $M_j(x) > 0$ if, and only if, $x_j < x < x_{j+n}$.

THEOREM 2.2. *Let $S(x)$ be a spline on (a,b) associated with $\mathcal{M}_n$, with simple knots $\{x_j\}_{j=-\infty}^{\infty}$, admitting the representation* (2.8). *Then*

$$S^-(S(x);(a,b)) \leqslant S^-(\{c_j\}_{j=-\infty}^{\infty}).$$

Proof. $M_j(x)$ is totally positive in j and x (see Theorem 4.1, Chapter 10, in [3]). Any totally positive kernel induces a variation-diminishing transformation (see Theorem 3.1, Chapter 5, in [3]).

SECTION 3. A VARIATION-DIMINISHING GENERALIZED SPLINE WITH AN INFINITE NUMBER OF KNOTS

We wish to find a variation-diminishing generalized spline associated with $\mathcal{M}_n$, with simple knote $\{x_j\}_{j=-\infty}^{\infty}$, which preserves generalized linear functions. Theorems 2.1 and 2.2 provide two of the key results.

We can regard $\phi_1(x)$ and $\phi_2(x)$ (see (1.3)) as splines associated with $\mathcal{M}_n$. Therefore, according to Theorem 2.1, there are unique representations

$$\phi_k(x) = \sum_{j=-\infty}^{\infty} a_j^{(k)} M_j(x), \qquad k = 1,2; a < x < b.$$

It will be useful to define

$$N_j(x) = d_j M_j(x), \tag{3.1}$$

where the d_j are positive constants, to be determined. In order to obtain the desired representation, we need to determine $\{d_j\}_{j=-\infty}^{\infty}$ and $\{z_j\}_{j=-\infty}^{\infty}$, $a < z_j < z_{j+1} < b$, such that

$$\frac{a_j^{(k)}}{d_j} = \phi_k(z_j), \qquad k = 1,2; -\infty < j < \infty; \tag{3.2}$$

and so we need

$$\frac{a_j^{(2)}}{a_j^{(1)}} = \frac{\phi_2(z_j)}{\phi_1(z_j)}, \qquad -\infty < j < \infty. \tag{3.3}$$

Since

$$\phi_2(x)/\phi_1(x) = \int_\alpha^x w_2(t)\, dt$$

is strictly increasing, in order to establish that $\{z_j\}$ is increasing, we must show that

$$\{a_j^{(2)}/a_j^{(1)}\} \tag{3.4}$$

is strictly increasing. Moreover, once $\{z_j\}$ is determined, we have

$$d_j = a_j^{(1)}/\phi_1(z_j), \qquad -\infty < j < \infty.$$

Therefore, we also wish to prove that

$$a_j^{(1)} > 0, \qquad -\infty < j < \infty. \tag{3.5}$$

In order to prove (3.4) and (3.5), we will establish the following more general result (the proof is given in Section 4).

THEOREM 3.1. *Let $\{M_j(x)\}_{j=-\infty}^{\infty}$ be the B-splines associated with $\mathcal{M}_n$ and the simple knots $\{x_j\}_{j=-\infty}^{\infty}$, as defined in Section 2. Let*

$$\phi_k(x) = \sum_{j=-\infty}^{\infty} a_j^{(k)} M_j(x), \qquad k = 1, 2, \ldots, n; a < x < b. \tag{3.6}$$

Then

$$\det \|a_{j_m}^{(l)}\|_{l,m=1}^{k} > 0, \qquad k = 1, 2, \ldots, n; -\infty < j_1 < j_2 < \ldots < j_k < \infty. \tag{3.7}$$

Schoenberg stated this result without proof for the case of polynomial splines in [6].

LEMMA 3.2. *When* (3.7) *holds, we can choose the nodes z_j in the interval (a, b).*

Proof. We will show that z_j can be determined satisfying (3.3) with $a < z_j$. The proof that $z_j < b$ is similar.

Obviously, we may assume $a > -\infty$. Suppose $a_i^{(2)}/a_i^{(1)} < \phi_2(a)/\phi_1(a)$ for some i. Then

$$\phi_2(x) = \sum_{j=i-n+1}^{i} \frac{a_j^{(2)}}{a_j^{(1)}} a_j^{(1)} M_j(x) < \frac{\phi_2(a)}{\phi_1(a)} \phi_1(x), \qquad x_i < x < x_{i+1}, \tag{3.8}$$

since $\phi_1(x) = \sum_{j=i-n+1}^{i} a_j^{(1)} M_j(x)$ when $x_i < x < x_{i+1}$ (recall that $M_j(x) \neq 0$ iff $x_j < x < x_{j+n}$), $a_j^{(1)} > 0$, and $a_j^{(2)}/a_j^{(1)}$ is strictly increasing. But (3.8) implies $\phi_2(x)/\phi_1(x) < \phi_2(a)/\phi_1(a)$, contradicting the fact that $\phi_2(x)/\phi_1(x)$ is strictly increasing.

Suppose that $f(x)$ is defined on (a,b). The generalized spline

$$S(x) = \sum_{j=-\infty}^{\infty} f(z_j) N_j(x) \tag{3.9}$$

is well-defined, where $N_j(x)$ is defined by (3.1) and (3.2), and z_j is defined by (3.3).

THEOREM 3.3. *The generalized spline approximation $S(x)$ defined in* (3.9) *preserves functions of the form $A\phi_1(x) + B\phi_2(x)$ (A and B constants) and is variation-diminishing on (a,b).*

Proof. We have defined the $N_j(x)$ and z_j in such a way that generalized linear functions are preserved. Since $a_j^{(1)} \phi_1(z_j) > 0$, we see from Theorem 2.2 that

$$S^-(S(x);(a,b)) \leqslant S^-(f(z_j);-\infty < j < \infty).$$

$\{z_j\}_{j=-\infty}^{\infty}$ is strictly increasing, so

$$S^-(f(z_j);-\infty < j < \infty) \leqslant S^-(f(x);(a,b)).$$

Remark. $\{z_j\}$ and $\{N_j(x)\}$ depend on the choice of α used as an initial-value point for $\phi_2(x)$.

SECTION 4. PROOF OF THEOREM 3.1

THEOREM 3.1. *Let $\{M_j(x)\}_{j=-\infty}^{\infty}$ be the basic spline functions associated with the operator $\mathcal{M}_n$ and simple knots $\{x_j\}_{j=-\infty}^{\infty}$, as defined in Section* 2. *Suppose that*

$$\phi_k(x) = \sum_{j=-\infty}^{\infty} a_j^{(k)} M_j(x), \qquad k = 1,2,\ldots,n; a < x < b. \tag{4.1}$$

Then

$$\det \|a_{j_m}^{(l)}\|_{l,m=1}^{k} > 0, \qquad k = 1,2,\ldots,n; -\infty < j_1 < j_2 < \ldots < j_k < \infty. \tag{4.2}$$

Remark. We must prove that the determinant of any $k \times k$ submatrix drawn from the first k rows of $\|a_j^{(l)}\|_{l=1,j=-\infty}^{n;\infty}$ is strictly positive. We will prove this result for submatrices composed of consecutive columns, and then use the Fekete theorem (Theorem 3.2 of Chapter 2 in [3]) to get (4.2). Since $w_1(x)$ is independent of the initial value point $x = \alpha$ for the fundamental solution set $\{\phi_j(x)\}_{j=1}^{n}((L_{i-1}\phi_j)(\alpha) = w_j(\alpha)\delta_{ij})$, it is easy to show that $\det\|\phi_i(x_j)\|_{i,j=1}^{p}$ is independent of the choice of α. Since $M_k(x)$ is also independent of the choice of α, we can assume that all B-splines are defined using the same initial value point.

Proof. We will need the following representation.

LEMMA 4.1. *For $j = 1, 2, \ldots, n$ and $x_n < s < x_{n+1}$,*

$$\hat{\phi}_n(x_{n+j}; s) = \sum_{i=1}^{j} c_i^{(j)} M_i(s), \tag{4.3}$$

where

$$c_j^{(j)} > 0. \tag{4.4}$$

Proof. When $x_n < s < x_{n+1}$, $\hat{\phi}_n(x_j; s) = 0$ for $j = 1, 2, \ldots, n$. Therefore

$$M_1(s) = \det \|\hat{\phi}_i(x_j)\|_{i,j=1}^{n} \cdot \hat{\phi}_n(x_{n+1}; s) / \det \|\hat{\phi}_i(x_j)\|_{i,j=1}^{n+1}.$$

Since $\det \|\hat{\phi}_i(x_j)\|_{i,j=1}^{n} > 0$ (see [3], Chapter 6, Theorem 1.1), the lemma is true for $j = 1$. The induction hypothesis is that (4.3) is true for $j = 1, 2, \ldots, k-1$, $2 \leqslant k \leqslant n$. Expanding $M_k(s)$, we get

$$M_k(s) = d_k \hat{\phi}_n(x_{n+k}; s) + \sum_{i=1}^{k-1} d_i \hat{\phi}_n(x_{n+i}; s),$$

$$d_k = \det \|\hat{\phi}_i(x_{k+j-1})\|_{i,j=1}^{n} / \det \|\hat{\phi}_i(x_{k+j-1})\|_{i,j=1}^{n+1}.$$

Since $d_k > 0$, it is clear using the induction hypothesis that (4.3) is valid for $j = k$.

In order to prove (4.2) for $n - k \times n - k$ submatrices composed of $n - k$ consecutive columns ($k = 0, 1, \ldots, n-1$), we consider the system of equations

$$\begin{aligned} \phi_i(s_j) &= \sum_{\mu=1}^{n} a_\mu^{(i)} M_\mu(s_j), \qquad i = 1, 2, \ldots, n-k; \\ \hat{\phi}_n(x_{n+l}; s_j) &= \sum_{\mu=1}^{l} c_\mu^{(l)} M_\mu(s_j), \qquad l = k, k-1, \ldots, 1, \end{aligned} \tag{4.5}$$

for $j = 1, 2, \ldots, n$, where the s_j are chosen so that $x_n < s_1 < s_2 < \ldots < s_n < x_{n+1}$ (recall that $M_k(x) \neq 0$ iff $x_k < x < x_{k+n}$). When $k = 0$, the equations for $\hat{\phi}_n(x_{n+l}; s_j)$ are omitted. In matrix form, (4.5) can be written

$$\left\| \begin{matrix} \|\phi_i(s_j)\|_{i,j=1}^{n-k,n} \\ \|\hat{\phi}_n(x_{n+l}; s_j)\|_{l=k,j=1}^{1,n} \end{matrix} \right\| = \left\| \begin{matrix} \|a_\mu^{(i)}\|_{i,\mu=1}^{n-k,n} \\ \|c_\mu^{(l)}\|_{l=k,\mu=1}^{1\ \ \ k} \ 0_{k,n-k} \end{matrix} \right\| \cdot \|M_\mu(s_j)\|_{\mu,j=1}^{n}, \tag{4.6}$$

where we define $c_j^{(i)} = 0$ when $i < j$, and $0_{k,n-k}$ is the $k \times n-k$ zero matrix. The determinant of the right-hand side of (4.6) is

$$(-1)^{\sum_{i=n-k+2}^{n+1} i} \left(\prod_{i=1}^{k} c_i^{(i)} \right) \cdot \det \|a_{k+m}^{(i)}\|_{i,m=1}^{n-k} \cdot \det \|M_\mu(s_m)\|_{\mu,m=1}^{n} \tag{4.7}$$

$$\overset{s}{=} (-1)^{\sum_{i=n-k+2}^{n+1} i} \det \|a_{k+m}^{(i)}\|_{i,m=1}^{n-k},$$

since $c_i^{(i)} > 0$ and $\det \|M_\mu(s_j)\|_{\mu,j=1}^n > 0$ (see [3], Chapter 10, Lemma 4.2); the symbol $c \overset{s}{=} d$ means that $cd > 0$.

In order to evaluate the determinant of the matrix on the left side of (4.6), we need the following representation, which is the non-self adjoint version of a representation formula in [2].

LEMMA 4.2.

$$\hat{\phi}_n(x;s) = \sum_{j=1}^{n} (-1)^{j-1} \hat{\phi}_{n+1-j}(x)\, \phi_j(s), \qquad s < x. \tag{4.8}$$

Proof. It is easy to see that $\mathscr{M}_n^{(s)} \hat{\phi}_n(x;s) \equiv 0$ for $s < x$ (the differentiations are to be performed with respect to s). Therefore, we can write

$$\hat{\phi}_n(x;s) = \sum_{j=1}^{n} c_j(x)\, \phi_j(s)$$

for $s < x$. In order to determine the coefficient of $\phi_j(s)$, operate on $\hat{\phi}_n(x;s)$ with $L_{j-1}^{(s)}$ (defined in (1.2)) and set $s = \alpha$. That (4.8) holds when $s < \alpha$, follows from the unicity of the initial-value problem for ordinary differential equations.

Let

$$a_j(x) = (-1)^{j-1} \hat{\phi}_{n+1-j}(x). \tag{4.9}$$

By using the representation (4.8), the determinant of the left side of (4.6) can be written as

$$\begin{vmatrix} \|\phi_i(s_j)\|_{i,j=1}^{n-k,n} \\ \|b_l(s_j)\|_{l=k,j=1}^{1,n} \end{vmatrix}, \tag{4.10}$$

where $b_l(s) = \sum_{t=1}^{n} a_t(x_{n+l})\, \phi_t(s)$. The matrix of (4.10) can be written in the form

$$\begin{Vmatrix} I_{n-k} & 0_k \\ \|a_j(x_{n+l})\|_{l=k,j=1}^{1,n} & \end{Vmatrix} \cdot \|\phi_i(s_j)\|_{i,j=1}^{n},$$

where I_{n-k} is the $n-k \times n-k$ identity matrix. Therefore, the determinant (4.10) is equal to

$$\det \|a_j(x_{n+l})\|_{l=k,j=n-k+1}^{1,n} \cdot \det \|\phi_i(s_j)\|_{i,j=1}^{n}. \tag{4.11}$$

According to the Remark above, we can assume that the initial value point α satisfies $\alpha < s_1$. Then $\det \|\phi_i(s_j)\|_{i,j=1}^n > 0$ by Theorem 1.1, Chapter 6 of [3]. According to (4.9), the first determinant in (4.11) is

$$\det \|(-1)^{n-j} \hat{\phi}_j(x_{n+l})\|_{l,j=k}^{1} = (-1)^{\sum_{j=n-k}^{n-1} j} \det \|\hat{\phi}_i(x_{n+l})\|_{i,j=1}^{k}$$
$$\overset{s}{=} (-1)^{\sum_{j=n-k}^{n-1} j}.$$

Comparing this with (4.7), we see that $\det \|a_j^{(i)}\|_{i=1,j=k+1}^{n-k,n} > 0$. Using a suitable translation, the same proof shows that

$$\det \|a_j^{(i)}\|_{i=1,j=m}^{k,m+k-1} > 0, \qquad k = 1,\ldots,n; -\infty < m < \infty. \tag{4.12}$$

To finish the proof, we remove the restriction that the columns be consecutive, by applying the Fekete theorem ([3], Chapter 2, Theorem 3.2) successively to (4.12), with $k = 1, k = 2, \ldots, k = n$.

SECTION 5. A VARIATION-DIMINISHING SPLINE FOR A FINITE INTERVAL WITH FINITELY-MANY KNOTS

As mentioned in Section 1, Schoenberg [6] pointed out that the key to finding a variation-diminishing polynomial spline with finitely many knots in (a,b), which also preserves generalized linear functions on $[a,b]$, is the introduction of knots of multiplicity n at $x = a$ and $x = b$. In this section we will define a generalized spline with these properties.

DEFINITION 5.1. Let $\{x_j\}$ satisfy $x_j < x_{j+1}$. $S(x)$ is a generalized spline with knots $\{x_j\}$, associated with the differential expression $\mathcal{M}_n$ (see (1.1)), if $(\mathcal{M}_n S)(x) = 0$, $x \neq x_j$. x_j is called a *knot of multiplicity* μ if

$$S(x) \in C^{n-1-\mu}[x_j - \epsilon, x_j + \epsilon]$$

for small positive ϵ.

A knot of multiplicity one is a simple knot, and $S(x)$ has a jump discontinuity at a knot of multiplicity n. See [3], Chapter 10, for more details.

We will want to consider continuous functions defined on a finite interval $[a,b]$ and approximating splines with m simple knots $\{x_j\}_{j=1}^m$ in (a,b), $a < x_1 < x_2 < \ldots < x_m < b$. We introduce knots of multiplicity n at $x = a$ and at $x = b$, so $S(x)$ has a jump discontinuity at these two points. We will assume that

$$S(x) = 0, \qquad x < a; x > b. \tag{5.1}$$

Set $x_0 = a$, $x_{m+1} = b$. Let $\{\hat{\phi}_j(x)\}_{j=1}^n$ be a basic set of solutions for $\hat{\mathcal{M}}_n u = 0$, as in (2.3), with initial values at $x = a$: $(\hat{L}_{i-1}\hat{\phi}_j)(a) = w_{n+2-j}(a)\delta_{ij}$.

The definition of the basic spline functions in Section 2 has to be modified for $k = 1, 2, \ldots, n-1$ and $k = m+2, m+3, \ldots, m+n$, as for these values $M_k(x)$ is a spline with multiple knots. Recall the definitions in (2.5) and (2.6). We define

$$M_k(x) = c_k \begin{vmatrix} \hat\phi_1(a) & \hat\phi_2(a) & \dots & \hat\phi_{n+1-k}(a) & \dots & \hat\phi_n(a) & \hat\phi_n(a;x) \\ 0 & \hat\phi_{2,2}(a) & \dots & \hat\phi_{2,n+1-k}(a) & \dots & \hat\phi_{2,n}(a) & \hat\phi_{2,n}(a;x) \\ \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & \hat\phi_{n+1-k,n+1-k}(a) & \dots & \hat\phi_{n+1-k,n}(a) & \hat\phi_{n+1-k,n}(a;x) \\ \hat\phi_1(x_1) & \hat\phi_2(x_1) & \dots & \hat\phi_{n+1-k}(x_1) & \dots & \hat\phi_n(x_1) & \hat\phi_n(x_1;a) \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ \hat\phi_1(x_k) & \hat\phi_2(x_k) & \dots & \hat\phi_{n+1-k}(x_k) & \dots & \hat\phi_n(x_k) & \hat\phi_n(x_k;a) \end{vmatrix},$$

$$k = 1,2,\dots,n-1, \qquad (5.2)$$

$$M_k(x) = c_k \begin{vmatrix} \hat\phi_1(x_{k-n}) & \hat\phi_2(x_{k-n}) & \dots & \hat\phi_{k-m}(x_{k-n}) & \dots & \hat\phi_n(x_{k-n}) & \hat\phi_n(x_{k-n};x) \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ \hat\phi_1(x_m) & \hat\phi_2(x_m) & \dots & \hat\phi_{k-m}(x_m) & \dots & \hat\phi_n(x_m) & \hat\phi_n(x_m;x) \\ \hat\phi_1(b) & \hat\phi_2(b) & \dots & \hat\phi_{k-m}(b) & \dots & \hat\phi_n(b) & \hat\phi_n(b;x) \\ 0 & \hat\phi_{2,2}(b) & \dots & \hat\phi_{2,k-m}(b) & \dots & \hat\phi_{2,n}(b) & \hat\phi_{2,n}(b;x) \\ \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & \hat\phi_{k-m,k-m}(b) & \dots & \hat\phi_{k-m,n}(b) & \hat\phi_{k-m,n}(b;x) \end{vmatrix},$$

$$k = m+2, m+3, \dots, m+n, \qquad (5.3)$$

where c_k is the reciprocal of the given determinant with the last column replaced by

$$(\hat\phi_{n+1}(a), \hat\phi_{2,n+1}(a), \dots, \hat\phi_{n+1-k,n+1}(a), \hat\phi_{n+1}(x_1), \dots, \hat\phi_{n+1}(x_k))$$

in (5.2), and in (5.3) by

$$(\hat\phi_{n+1}(x_{k-n}), \dots, \hat\phi_{n+1}(x_m), \hat\phi_{n+1}(b), \hat\phi_{2,n+1}(b), \dots, \hat\phi_{k-m,n+1}(b)).$$

For the remaining values of k, $M_k(x)$ is defined as is $M_{k-n}(x)$ in Section 2. As in Section 2, $c_k > 0$. (When $m+2-n<0$, modifications as in (5.2) and (5.3) must be made, in both the upper and lower parts of the determinants defining some of the basic spline functions; see [*4*] for details.)

For these basic spline functions with multiple knots, results analogous to those in Section 2 are valid.

THEOREM 5.1. *Let $S(x)$ be a spline associated with the differential expression $\mathcal{M}_n$, with knots $\{x_j\}_{j=0}^{m+1}$, $x_j < x_{j+1}$, where x_j is a knot of multiplicity μ_j, $1 \leqslant \mu_j \leqslant n$. If $\mu = \sum_{j=0}^{m+1} \mu_j \geqslant n+1$, and $S(x) = 0$ for $x \notin [x_0, x_{m+1}]$, then $S(x)$ can be represented uniquely in the form*

$$S(x) = \sum_{j=1}^{\mu-n} c_j M_j(x), \qquad (5.4)$$

where $\{M_j(x)\}$ is the set of basic spline functions for the given knots with the given multiplicities.

Proof. This theorem involves a straightforward generalization of Theorem 4.2, Chapter 10, of [3]; it was stated for the polynomial spline case in [1].

Therefore, there are unique representations

$$\phi_k(x) = \sum_{j=1}^{m+n} a_j^{(k)} M_j(x), \qquad a \leqslant x \leqslant b; k = 1,2,\ldots,n, \tag{5.5}$$

where the $\phi_k(x)$ are as defined in (1.3) for $a \leqslant x \leqslant b$ and zero outside $[a,b]$, and the $M_j(x)$ are the B-splines associated with $\mathscr{M}_n$ and the knots $\{x_j\}_{j=0}^{m+1}$, as defined above. As in Section 3 for the case of an infinite number of knots, define

$$N_j(x) = d_j M_j(x), \qquad j = 1,2,\ldots,m+n, \tag{5.6}$$

where the d_j are positive constants, to be determined. In order to obtain the desired representation, we need to determine $\{d_j\}_{j=1}^{m+n}$ and $\{z_j\}_{j=1}^{m+n}$, $a \leqslant z_j < z_{j+1} \leqslant b$, such that

$$a_j^{(k)}/d_j = \phi_k(z_j), \qquad k = 1,2; j = 1,2,\ldots,m+n;$$

so we need

$$a_j^{(2)}/a_j^{(1)} = \phi_2(z_j)/\phi_1(z_j), \qquad j = 1,2,\ldots,m+n. \tag{5.7}$$

As in Section 3, it is sufficient to show that $a_j^{(1)} > 0$, $a_j^{(2)}/a_j^{(1)}$ is strictly increasing in j, and z_1, $z_{m+n} \in [a,b]$.

THEOREM 5.2. *Let $\{M_j(x)\}_{j=1}^{m+n}$ be the B-splines associated with $\mathscr{M}_n$ and the knots $\{x_j\}_{j=0}^{m+1}$, as defined above. With $a_j^{(k)}$ defined as in* (5.5),

$$\det \|a_{j_m}^{(l)}\|_{l,m=1}^{k} > 0; \qquad k = 1,2,\ldots,n; 1 \leqslant j_1 < j_2 < \ldots < j_k \leqslant m+n.$$

Schoenberg stated this result for polynomial splines in [6], but the proof has not been published. One shows that

$$\det \|a_{j+r-1}^{(l)}\|_{l,j=1}^{k} > 0 \qquad \text{for } k = 1,2,\ldots,n; r = 1,2,\ldots,n+m-k+1, \tag{5.8}$$

and then uses the Fekete theorem. However, if $1 \leqslant k < n$ and $1 \leqslant r \leqslant n-k$, there is no $n \times n$ submatrix with the matrix in (5.8) in the upper-right corner. We can get our hands on the matrix in (5.8) by considering the system of equations

$$\phi_i(s) = \sum_{j=1}^{n} a_j^{(i)} M_j(s), \qquad i = 1,2,\ldots,p,$$

$$\hat{\phi}_n(x_{n+l}; s) = \sum_{j=1}^{l} c_j^{(l)} M_j(s), \qquad l = q, q-1,\ldots,1,$$

$$M_{n-t+1}(s) = M_{n-t+1}(s), \qquad t = r, r-1,\ldots,1,$$

where $a = x_0 < x < x_1$, and p, q, and r are non-negative integers such that $p+q+r=n$, $p \geqslant 1$. A few technical variations must be made in the method used to prove Theorem 3.1; see [4] for details.

LEMMA 5.3. *We can define* $\{z_j\}_{j=1}^{m+n}$ *as in* (5.7), *with* $z_1 = a$, $z_{m+n} = b$.

Proof. We can write $M_k(x) = \lim_{t \downarrow a} M_k(x;t)$, $1 \leqslant k \leqslant n$, where $M_k(x;t)$ is defined similarly to $M_k(x)$, but with a replaced by t in the numerator. Since $M_k(x;t) \geqslant 0$, with strict inequality if and only if $t < x < x_k$ (see Theorem 1.1, Chapter 10, of [3]), $M_k(x) = 0$ unless $a < x < x_k$. $M_k(x)$ has a knot of multiplicity $n+1-k$ at $x=a$ for $1 \leqslant k \leqslant n$. Therefore, $M_k(x)$ is continuous at $x=a$ for $2 \leqslant k \leqslant n$, so $M_k(x) \to 0$ as $x \downarrow a$ for these values of k. From the definition, it is clear that $M_1(x) \to c\phi_n(x_1)$ as $x \downarrow a$, $c \neq 0$. Therefore

$$0 = \lim_{x \downarrow a} \phi_2(x) = a_1^{(2)} c\hat{\phi}_n(x_1),$$

so we must have $a_1^{(2)} = 0$. Thus, if we define z_j by (5.7), $z_1 = a$.

It is easy to see that $M_j(x) \to 0$ as $x \uparrow b$ for $j = 1, 2, \ldots, m+n-1$. Therefore,

$$\phi_2(b) = \lim_{x \uparrow b} \frac{a_{m+n}^{(2)}}{a_{m+n}^{(1)}} a_{m+n}^{(1)} M_{m+n}(x),$$

$$\phi_1(b) = \lim_{x \uparrow b} a_{m+n}^{(1)} M_{m+n}(x).$$

From these equations we see that (5.7) is valid for $j = m+n$ if we choose $z_{m+n} = b$.

Let $\{z_j\}_{j=1}^{m+n}$ be defined by (5.7). We have shown that $z_j \in [a,b]$, and the z_j are strictly increasing. Define

$$N_j(x) = a_j^{(1)} M_j(x)/\phi_1(z_j), \qquad j = 1, 2, \ldots, m+n.$$

We consider the generalized spline approximation

$$S(x) = \sum_{j=1}^{m+n} f(z_j) N_j(x), \qquad a \leqslant x \leqslant b. \tag{5.9}$$

THEOREM 5.4. *The generalized spline approximation method defined in* (5.9) *is variation-diminishing on* $[a,b]$ *and preserves functions of the form*

$$A\phi_1(x) + B\phi_2(x).$$

Proof. The $N_j(x)$ and z_j have been chosen so that generalized linear functions are preserved. It can be shown, as in [3], Chapter 10, that $M_j(x)$ is totally positive in j and x, therefore $N_j(x)$ is also so. By the argument used in the proof of Theorem 3.3, this implies that the transformation in (5.9) is variation-diminishing.

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